

# Functional equations and Nemytskij operator

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**Abstract.** In this chapter we prove that a functional equation of the form

$$\phi(x) = g(x, \phi[f(x)])$$

has a unique solution under some assumptions and consider the problem of determining it (for that we use a classical theorem due to J. Matkowski). Furthermore, we prove the set-valued analogue of Matkowski's result.

**Keywords:** Nemytskij operator,  $\text{Lip}^\alpha$  space, set-valued functions, functional equations, Jensen equation.

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## Introduction

The following chapter consists of two sections. In the first section we consider a functional equation of the form

$$\phi(x) = g(x, \phi[f(x)]).$$

We apply the Schauder Fixed Point Theorem to show that this equation has a solution under some conditions. Moreover, we consider the problem of determining its solution with the help of the Banach Fixed Point Theorem. We prove that the Banach Principle can be applied only in the linear case. The crucial result on which the proof of this fact is based on, is a classical theorem due to J. Matkowski. In 1982 he showed (cf. [11]), that a Nemytskij operator  $N$  mapping the function space  $\text{Lip}(I, \mathbb{R})$  into itself is Lipschitzian with respect to the Lipschitzian norm if and only if its generator is of the form

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$$g(x, y) = a(x)y + b(x), \quad x \in I, y \in \mathbb{R},$$

for some  $a, b \in \text{Lip}(I, \mathbb{R})$ . This result was extended to many spaces by J. Matkowski and others (cf. e.g. [12]). Set-valued versions of Matkowski's results were investigated for instance in papers [3, 6–9, 15, 17, 21]. Recently Matkowski has shown (cf. [14]), that if we only assume, that the operator  $N$  is uniformly continuous, then the generator  $g$  is of the form above.

The second part of this chapter contains results from a paper by E. Mainka (cf. [9]). The main goal of it is to prove the set-valued analogue of Matkowski's result for superposition operators mapping the set  $H_\alpha(I, C)$  of all Hölder functions  $\varphi: I \rightarrow C$  into the set  $H_\beta(I, \text{clb}(Z))$  of all Hölder set-valued functions  $\phi: I \rightarrow \text{clb}(Z)$ .

## 1. Fixed Point Theorems and Nemytskij operators

In the following we shall write  $I$  for a unit interval  $[0, 1]$  on the real line. If  $E, E'$  are nonempty sets, then let us denote by  $\mathcal{F}(E, E')$  the set consisting of all maps from  $E$  into  $E'$ . Real bounded functions defined on interval  $I$  form a Banach space  $B(I, \mathbb{R})$  with the uniform convergence norm  $\|\cdot\|_{B(I, \mathbb{R})}$ . Real continuous functions defined on  $I$  form closed linear subspace  $C(I, \mathbb{R})$  of  $B(I, \mathbb{R})$ , therefore it is a Banach space. The uniform convergence norm in it we denote by  $\|\cdot\|_{C(I, \mathbb{R})}$ . Moreover, the set  $\text{Lip}(I, \mathbb{R})$  of all real functions defined on  $I$  and satisfying the Lipschitz condition with the norm given by

$$\|\phi\|_{\text{Lip}(I, \mathbb{R})} := |\phi(0)| + \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|}, \quad \phi \in \text{Lip}(I, \mathbb{R})$$

is also a Banach space.

Now let us consider functional equation

$$\phi(x) = g(x, \phi[f(x)]), \tag{1}$$

where  $g: I \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f: I \rightarrow I$  are given functions. We seek for the solution  $\phi: I \rightarrow \mathbb{R}$  of equation (1).

**Theorem 1.1.** *Assume that a function  $f: I \rightarrow I$  satisfies the Lipschitz condition*

$$|f(x_1) - f(x_2)| \leq s|x_1 - x_2|, \quad x_1, x_2 \in I \tag{2}$$

*with the constant  $0 < s < 1$ . Let  $f(0) = 0$  and let  $g: I \times \mathbb{R} \rightarrow \mathbb{R}$  be a function for which the inequality*

$$|g(x_1, y_1) - g(x_2, y_2)| \leq p|x_1 - x_2| + q|y_1 - y_2|, \quad x_1, x_2 \in I, y_1, y_2 \in \mathbb{R}, \tag{3}$$

*holds for  $p, q > 0$  and let  $sq < 1$ . Moreover, let  $d \in \mathbb{R}$  be a fixed point of the function  $g(0, \cdot)$ , i.e.*

$$d = g(0, d). \tag{4}$$

*Then there exists exactly one solution of equation (1) in the class  $\text{Lip}(I, \mathbb{R})$ , for which  $\phi(0) = d$ .*

*Proof.* Let  $\Phi_0$  be a subset of  $C(I, \mathbb{R})$  consisting of all the functions  $\phi$ , for which  $\phi(0) = d$  and which satisfy the Lipschitz condition with a constant  $k$ , where

$$k := p/(1 - sq). \tag{5}$$

It is easy to see that  $\Phi_0$  is a convex and uniformly closed subset of  $C(I, \mathbb{R})$ . Moreover, functions from the set  $\Phi_0$  satisfy Lipschitz condition with a fixed constant (equal to  $k$ ). Hence  $\Phi_0$  is uniformly bounded and constitutes an equicontinuous family of functions. Thus Arzela-Ascoli theorem implies that  $\Phi_0$  is a compact subset of the space  $C(I, \mathbb{R})$ . Now, let us define a map  $T: \Phi_0 \rightarrow \mathcal{F}(I, \mathbb{R})$  as follows:

$$T(\phi)(x) := g(x, \phi[f(x)]), \quad \phi \in \Phi_0, x \in I.$$

We are going to show that the range of  $T$  is a subset of  $\Phi_0$ . Let  $\phi \in \Phi_0$  and let  $x_1, x_2 \in I$ . From (5) and from inequalities (2) and (3) we get

$$\begin{aligned} |T(\phi)(x_1) - T(\phi)(x_2)| &= |g(x_1, \phi[f(x_1)]) - g(x_2, \phi[f(x_2)])| \leq \\ &\leq p|x_1 - x_2| + q|\phi[f(x_1)] - \phi[f(x_2)]| \leq p|x_1 - x_2| + qks|x_1 - x_2| \leq \\ &\leq (p + qks)|x_1 - x_2| = k|x_1 - x_2|, \end{aligned}$$

thus  $T(\phi)$  satisfies the Lipschitz condition with the constant  $k$ . Moreover, from (4) it follows that

$$T(\phi)(0) = g(0, \phi[f(0)]) = g(0, \phi(0)) = g(0, d) = d,$$

which finishes the proof of the fact, that  $T$  is a self-map of the set  $\Phi_0$ . Now we are going to show that  $T$  is continuous. For this, assume that  $\phi_1, \phi_2 \in \Phi_0$ . From inequality (3) we get

$$\begin{aligned} |T(\phi_1)(x) - T(\phi_2)(x)| &= |g(x, \phi_1[f(x)]) - g(x, \phi_2[f(x)])| \leq q|\phi_1[f(x)] - \phi_2[f(x)]| \leq \\ &\leq q \sup_{x \in I} |\phi_1(x) - \phi_2(x)| = q\|\phi_1 - \phi_2\|_{C(I, \mathbb{R})} \end{aligned}$$

and in consequence

$$\|T(\phi_1) - T(\phi_2)\|_{C(I, \mathbb{R})} \leq q\|\phi_1 - \phi_2\|_{C(I, \mathbb{R})},$$

which implies the continuity of  $T$ . From Schauder Theorem we infer that  $T$  has a fixed point  $\phi_0$ , which, on account of the definition of the set  $\Phi_0$ , is a lipschitzian solution of (1) satisfying the equality  $\phi_0(0) = d$ . It remains to show that these conditions determine a solution uniquely.

Let  $\phi_1$  and  $\phi_2$  be solutions of equation (1), for which  $\phi_1(0) = \phi_2(0) = d$  and which satisfy the Lipschitz condition with a constant  $L$ . Let us define  $Q_1, Q_2: I \rightarrow \mathbb{R}$  as follows:

$$Q_i(x) := \begin{cases} (1/x)[\phi_i(x) - d] & \text{for } x \in (0, 1], \\ 0 & \text{for } x = 0, \end{cases}$$

for  $i = 1, 2$  and let us note, that  $\phi_i(x) = d + xQ_i(x), x \in I$ . Since  $\phi_1$  and  $\phi_2$  satisfy the Lipschitz condition with the constant  $L$ , we obviously have  $|Q_i(x)| \leq L, i = 1, 2, x \in I$ . Thus (1) implies that

$$d + Q_i(x)x = g(x, d + f(x)Q_i[f(x)]).$$

Now define a map  $h: (0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  in the following way

$$h(x, y) := \frac{1}{x}[g(x, d + yf(x)) - d]$$

and let us note, that for  $x \in (0, 1]$  we have

$$h(x, Q_i[f(x)]) = \frac{1}{x}[g(x, d + f(x)Q_i[f(x)]) - d] = Q_i(x). \quad (6)$$

Let  $x \in (0, 1], y_1, y_2 \in \mathbb{R}$ . From (2), (3) and from the inclusion  $f(I) \subseteq I$  we get

$$\begin{aligned} |h(x, y_1) - h(x, y_2)| &= \frac{1}{x}|g(x, d + y_1f(x)) - g(x, d + y_2f(x))| \leq \\ &\leq \frac{1}{x}qf(x)|y_1 - y_2| \leq \frac{1}{x}q|f(x) - f(0)||y_1 - y_2| = qs|y_1 - y_2|. \end{aligned} \quad (7)$$

Thus, on account of (7) we get for  $x \in (0, 1]$

$$\begin{aligned} |Q_1(x) - Q_2(x)| &= |h(x, Q_1[f(x)]) - h(x, Q_2[f(x)])| \leq \\ &\leq qs|Q_1[f(x)] - Q_2[f(x)]| \leq qs\|Q_1 - Q_2\|_{B(I, \mathbb{R})}. \end{aligned}$$

Taking the supremum over  $x \in [0, 1]$  we obtain

$$\|Q_1 - Q_2\|_{B(I, \mathbb{R})} \leq qs\|Q_1 - Q_2\|_{B(I, \mathbb{R})}.$$

Since we assumed that  $qs < 1$ , we have  $Q_1 = Q_2$  and as a consequence we get  $\phi_1 = \phi_2$ , which finishes the proof.  $\square$

Theorem 1.1 with local Lipschitz conditions on the functions  $f$  and  $g$  and with assumption  $f(x) > 0$  for  $x > 0$  is formulated in J. Matkowski's work [10] (see also [5, Theorem 5.5.1, p. 205]). In the proof of Theorem 1.1 we obtained the existence of a lipschitzian solution of the equation (1) from the Schauder Fixed Point Theorem. Independently, we proved that such a solution is unique. Thus, in this situation it seems likely, that Theorem 1.1 could be proved with the help of the Banach Fixed Point Theorem. We are going to show, that the properties of the Nemytskij operator enable us to formulate the following conclusion: it is possible to apply the Banach Fixed Point Theorem only for the linear equation of type (1) (cf. [11] and [5, p. 206–209]).

Let  $E$  and  $E'$  be any given non-empty sets and let  $g: I \times E \rightarrow E'$  be a given function. We shall say, that  $g$  generates the *Nemytskij operator*

$$N: \mathcal{F}(I, E) \rightarrow \mathcal{F}(I, E'),$$

defined in the following way

$$(N\phi)(x) := g(x, \phi(x)), \quad \phi \in \mathcal{F}(I, E), x \in I. \quad (8)$$

Let us again consider (this time in the context of the definition of the Nemytskij operator), the functional equation of the form

$$\phi(x) = g(x, \phi[f(x)]), \tag{9}$$

where  $g: I \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f: I \rightarrow I$ . Let us note that if we define  $S$  by

$$S(\phi) := \phi \circ f, \quad \phi \in \mathcal{F}(I, \mathbb{R}),$$

then this definition enables us to write equation (9) in the following form

$$\phi = (N \circ S)\phi.$$

Assume now, that  $f: I \rightarrow I$  is a lipschitzian function. It is easy to see, that the composition  $\phi \circ f (= S(\phi))$  is an element of the space  $\text{Lip}(I, \mathbb{R})$  for every  $\phi \in \text{Lip}(I, \mathbb{R})$ . Thus  $S$  is a self-map of the space  $\text{Lip}(I, \mathbb{R})$ . Moreover,  $S$  is a continuous, linear operator on the Banach space  $\text{Lip}(I, \mathbb{R})$ . We shall now prove continuity. Assume that  $f(0) > 0$ . First, let us note that

$$|\phi(f(0))| \leq |\phi(0)| + |\phi(f(0)) - \phi(0)| = |\phi(0)| + \frac{|\phi(f(0)) - \phi(0)|}{|f(0) - 0|} f(0).$$

Thus

$$|\phi(f(0))| \leq |\phi(0)| + f(0) \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|}$$

and the obtained inequality is also true in the case  $f(0) = 0$ . Moreover, let us note that for  $t_1, t_2 \in I$ , for which the inequality  $f(t_1) \neq f(t_2)$  is satisfied, we have

$$\begin{aligned} \frac{|\phi(f(t_1)) - \phi(f(t_2))|}{|t_1 - t_2|} &= \frac{|\phi(f(t_1)) - \phi(f(t_2))|}{|f(t_1) - f(t_2)|} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|} \leq \\ &\leq \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \end{aligned}$$

and the inequality (obtained, on account of the assumption  $t_1, t_2 \in I, f(t_1) \neq f(t_2)$ ):

$$\frac{|\phi(f(t_1)) - \phi(f(t_2))|}{|t_1 - t_2|} \leq \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is obviously true also in the case  $f(t_1) = f(t_2)$ . We conclude, that

$$\sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(f(x_1)) - \phi(f(x_2))|}{|x_1 - x_2|} \leq \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} \sup_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}.$$

Thus

$$\|S(\phi)\|_{\text{Lip}(I, \mathbb{R})} = \|\phi \circ f\|_{\text{Lip}(I, \mathbb{R})} = |\phi(f(0))| + \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in I}} \frac{|\phi(f(x_1)) - \phi(f(x_2))|}{|x_1 - x_2|} =$$

$$\begin{aligned}
 &= |\phi(0)| + \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in I}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} \left[ f(0) + \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in I}} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \right] = \\
 &= |\phi(0)| + \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in I}} \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|} \|f\|_{\text{Lip}(I, \mathbb{R})} \leq M \|\phi\|_{\text{Lip}(I, \mathbb{R})},
 \end{aligned}$$

where  $M := \max\{1, \|f\|_{\text{Lip}(I, \mathbb{R})}\}$ , from which we infer the continuity of  $S$ .

Now we are going to show that  $f$  is a surjective self-map of the interval  $I$  if and only if  $S$  maps the space  $\text{Lip}(I, \mathbb{R})$  injectively into itself. Assume first, that  $f(I) = I$  and consider functions  $\phi, \psi \in \text{Lip}(I, \mathbb{R})$  for which  $S(\phi) = S(\psi)$ . The definition of  $S$  implies, that  $\phi(f(x)) = \psi(f(x))$  for  $x \in I$ . On account of surjectivity of  $f$ , every element  $y$  in the interval  $I$  is of the form  $y = f(x)$  and we infer that  $\phi(y) = \psi(y)$  for every  $y \in I$ , which finishes the proof of the injectivity of the map  $S$ .

Conversely, assume that  $f(I) \subseteq I, f(I) \neq I$ . Continuity of  $f$  (which is a consequence of the Lipschitz condition) and compactness of its domain imply that  $f$  is bounded and

$$f(I) = [m, M],$$

where

$$m := \inf_{x \in I} f(x), M := \sup_{x \in I} f(x).$$

The assumption  $f(I) \subseteq I, f(I) \neq I$  implies that at least one of the inequalities  $0 < \inf_{x \in I} f(x), \sup_{x \in I} f(x) < 1$  hold. Let us define  $\tilde{f}: I \rightarrow \mathbb{R}$  by

$$\tilde{f}(x) = \begin{cases} \frac{f(m)}{m}x & \text{for } x \in [0, m], \text{ if } 0 < m, \\ f(x) & \text{for } x \in [m, M](= f(I)), \\ \frac{1-f(M)}{1-M}(x - M) + f(M) & \text{for } x \in [M, 1], \text{ if } M < 1. \end{cases}$$

Thus  $\tilde{f}$  is an affine function on the interval  $[0, m]$  (if  $0 < m$ ) and on the interval  $[M, 1]$  (if  $M < 1$ ). Moreover, it satisfies conditions  $\tilde{f}(0) = 0, \tilde{f}(1) = 1$  and it is a continuous extension of the function  $f|_{f(I)}$  to the interval  $I$ . It is also obvious, that  $\tilde{f}$  satisfies the Lipschitz condition. From the definition of the function  $\tilde{f}$  we infer, that for  $x \in I$  the equality  $\tilde{f}(f(x)) = f(f(x))$  holds. Thus  $\tilde{f} \circ f = f \circ f$  and we obtain equality  $S(\tilde{f}) = S(f)$ . Moreover, let us note that at least one of the inequalities  $0 < f(0), f(1) < 1$  holds (since  $f(I) \neq I$ ), which, along with the equalities  $\tilde{f}(0) = 0, \tilde{f}(1) = 1$  implies, that  $\tilde{f} \neq f$ . Thus, the equality  $S(\tilde{f}) = S(f)$  enables us to conclude, that  $S$  is not injective.

Now, assume that  $f(I) = I$  and let  $S$  be a surjective map of the space  $\text{Lip}(I, \mathbb{R})$  onto itself. Thus  $S$  is a continuous and linear bijection, for which the inverse map (on account of the Banach Open Mapping Theorem) is also continuous. From surjectivity of  $S$  we obtain that for every function  $\phi \in \text{Lip}(I, \mathbb{R})$  there exists a function  $\psi \in \text{Lip}(I, \mathbb{R})$ , for which

$$\phi(x) = \psi(f(x))(= S(\psi)), \quad x \in I.$$

In particular, for a function  $\phi$  given by  $\phi(x) = x$  for  $x \in I$ , there exists a function  $\psi_0 \in \text{Lip}(I, \mathbb{R})$ , for which the equality

$$\psi_0(f(x)) = x$$

holds for  $x \in I$ . From this equality we infer that  $f$  is injective (and, in consequence, it is a bijection), and that  $f^{-1} = \psi_0$ , which implies that  $f^{-1}$  satisfies the Lipschitz condition.

Let us assume once again that  $\phi$  and  $\psi$  belong to the space  $\text{Lip}(I, \mathbb{R})$  and let the equality  $S(\psi) = \phi$  hold. Hence  $\phi(x) = \psi(f(x))$  for  $x \in I$ , which implies that  $\psi(y) = \phi(f^{-1}(y))$  for  $y \in I$ . Thus  $S^{-1}(\phi) = \phi \circ f^{-1}$  and this equality is satisfied for every function  $\phi \in \text{Lip}(I, \mathbb{R})$ .

Conversely, if  $f$  is bijective and lipschitzian function on  $I$ , for which  $f^{-1}$  is also lipschitzian, then  $S$  is a bijective self-map of the space  $\text{Lip}(I, \mathbb{R})$ , the equality  $S^{-1}(\phi) = \phi \circ f^{-1}$  holds and the map  $S^{-1}$  is continuous.

Assume that we are trying to apply the Banach Fixed Point Theorem to equation (9). For this, we have to assume that  $N \circ S$  is a contraction (with a constant  $k < 1$ ) of the space  $\text{Lip}(I, \mathbb{R})$  (with the lipschitzian norm). Let  $\phi \in \text{Lip}(I, \mathbb{R})$ . Then  $S^{-1}(\phi) \in \text{Lip}(I, \mathbb{R})$  and  $(N \circ S)(S^{-1}(\phi)) \in \text{Lip}(I, \mathbb{R})$ , since we are assuming that  $N \circ S$  maps the space  $\text{Lip}(I, \mathbb{R})$  into itself. Also  $(N \circ S)(S^{-1}(\phi)) = N\phi$  and we infer that  $N$  maps the space  $\text{Lip}(I, \mathbb{R})$  into itself. Now, let  $\phi_1, \phi_2 \in \text{Lip}(I, \mathbb{R})$ . Since  $N \circ S$  is a contraction, we obtain

$$\begin{aligned} \|N\phi_1 - N\phi_2\|_{\text{Lip}(I, \mathbb{R})} &\leq \|(N \circ S)(S^{-1}(\phi_1)) - (N \circ S)(S^{-1}(\phi_2))\|_{\text{Lip}(I, \mathbb{R})} \leq \\ &\leq k\|S^{-1}(\phi_1 - \phi_2)\|_{\text{Lip}(I, \mathbb{R})} \leq k\|S^{-1}\|_{L(\text{Lip}(I, \mathbb{R}))}\|\phi_1 - \phi_2\|_{\text{Lip}(I, \mathbb{R})}. \end{aligned}$$

Thus we infer, that  $N$  satisfies the Lipschitz condition. Now we shall quote the following Theorem of J. Matkowski [11] (see also [5, Theorem 5.5.2, p. 207]).

**Theorem 1.2.** *Let  $N$  be a Nemytskij operator generated by a function  $g: I \times \mathbb{R} \rightarrow \mathbb{R}$ . Conditions*

- 1)  $N: \text{Lip}(I, \mathbb{R}) \rightarrow \text{Lip}(I, \mathbb{R})$ ,
- 2) *there exists a constant  $L \geq 0$  such that*

$$\|N\phi_1 - N\phi_2\|_{\text{Lip}(I, \mathbb{R})} \leq L\|\phi_1 - \phi_2\|_{\text{Lip}(I, \mathbb{R})}, \quad \phi_1, \phi_2 \in \text{Lip}(I, \mathbb{R})$$

*are simultaneously satisfied if and only if there exist functions  $a, b \in \text{Lip}(I, \mathbb{R})$  for which*

$$g(x, y) = a(x)y + b(x), \quad x \in I, y \in \mathbb{R}.$$

This theorem enables us to conclude, that if the operator  $N \circ S$  is a contraction, then (given that  $f$  is bijective and lipschitzian along with its inverse)  $N\phi(x) = a(x)\phi(x) + b(x)$ , where  $a, b \in \text{Lip}(I, \mathbb{R})$ . Thus, in the nonlinear case it is not possible to apply the Banach Fixed Point Theorem to investigate the solvability of equation (1). Theorem 1.2 implies also, that it is possible to apply the Banach Principle to determine the lipschitzian solution of the linear equation of the form  $\phi(x) = a(x)\phi[f(x)] + b(x)$ .

## 2. Uniformly continuous Nemytskij operators

For given intervals  $I, J \subset \mathbb{R}$  and numbers  $\alpha \in (0, 1]$ ,  $x_0 \in I$  let us define the set  $\text{Lip}^\alpha(I, J)$  of all functions  $\varphi: I \rightarrow J$  for which the set

$$\left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} : x, y \in I, x \neq y \right\}$$

is bounded with the functional

$$\|\varphi\|_{\text{Lip}^\alpha} = |\varphi(x_0)| + \sup_{x, y \in I, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}. \quad (10)$$

In [14] Matkowski has shown that if a uniformly continuous with respect to the norm (10) superposition operator  $N$  of a generator  $f$  maps the set  $\text{Lip}^\alpha(I, J)$  into the Banach space  $\text{Lip}^\alpha(I, \mathbb{R})$ , then for some  $a, b \in \text{Lip}^\alpha(I, \mathbb{R})$  we have

$$f(x, y) = a(x)y + b(x), \quad x \in I, y \in J.$$

Our main goal is to prove a counterpart of Matkowski's result for Nemytskij operators generated by set-valued functions with values in a set  $\text{clb}(Z)$  of all nonempty, bounded, closed, convex subsets of a normed linear space  $Z$ .

Let  $(Z, \|\cdot\|)$  be a real, normed linear space. For a bounded  $A \subset Z$  one can define a number  $\|A\|$  as follows  $\|A\| := \sup\{\|z\| : z \in A\}$ .

By  $+$  we denote a binary operation in  $\text{clb}(Z)$  defined by the formula

$$A +^* B = \text{cl}(A + B),$$

where  $A + B$  is an algebraic sum of  $A$  and  $B$  and  $\text{cl}A$  is the closure of  $A$ . Note, that for arbitrary  $A, B \in \text{clb}(Z)$  the set  $A + B$  does not have to be closed. A corresponding example can be found e.g. in [20]. The pair  $(\text{clb}(Z), +^*)$  is an Abelian semigroup with the set  $\{0\}$  as the zero element. We can multiply elements of  $\text{clb}(Z)$  by nonnegative numbers and the conditions

$$1 \cdot A = A, \quad \lambda(\mu A) = (\lambda\mu)A, \quad \lambda(A +^* B) = \lambda A +^* \lambda B, \quad (\lambda + \mu)A = \lambda A +^* \mu A$$

hold for all  $A, B \in \text{clb}(Z)$  and  $\lambda, \mu \geq 0$ . This means that the set  $\text{clb}(Z)$  with operations  $+$  and  $\cdot$  is an *abstract convex cone*. The cancellation law, i.e.

$$A +^* B = C +^* B \implies A = C$$

in  $\text{clb}(Z)$  follows e.g. from Theorem II-17 in [2, p. 48].

It is easy to check that  $(\text{clb}(Z), d)$  is a metric space. It is complete, provided  $Z$  is a Banach space (cf. e.g. [2, p. 40]). It is easily seen that the Hausdorff distance is invariant with respect to translation, i.e.,

$$d(A +^* B, C +^* B) = d(A + B, C + B) = d(A, C)$$



(cf. e.g. [4]) and

$$d(\lambda A, \lambda B) = \lambda d(A, B)$$

for all  $\lambda \geq 0$  and  $A, B, C \in clb(Z)$ .

We say that a subset  $C$  of a real linear space  $Y$  is a *convex cone* if  $\lambda C \subset C$  for all  $\lambda \geq 0$  and  $C + C \subset C$ .

A set-valued function  $F: C \rightarrow clb(Z)$  defined on a convex cone  $C$  is *\*additive* (*\*Jensen*) if

$$F(x + y) = F(x) \overset{*}{+} F(y) \quad \left( F\left(\frac{x + y}{2}\right) = \frac{1}{2}(F(x) \overset{*}{+} F(y)) \right)$$

for all  $x, y \in C$ . A function  $F$  is  $\mathbb{Q}_+$ -homogeneous if  $F(\lambda y) = \lambda F(y)$  for all  $\lambda \in \mathbb{Q} \cap [0, \infty)$  and  $y \in C$ . We shall need the following lemmas.

**Lemma 2.1** (Corollary 4 in [19]). *Let  $C$  be a convex cone in a real linear space  $Y$  and let  $Z$  be a Banach space. A set-valued function  $F: C \rightarrow clb(Z)$  is \*Jensen if and only if there exist a \*additive set-valued function  $A: C \rightarrow clb(Z)$  and a set  $B \in clb(Z)$  such that*

$$F(x) = A(x) \overset{*}{+} B$$

for all  $x \in C$ .

**Lemma 2.2** (Lemma 2 in [16]). *Let  $Y, Z$  be two real, normed linear spaces and let  $C$  be a convex cone in  $Y$ . Suppose  $F$  is a  $\mathbb{Q}_+$ -homogeneous set-valued function defined on  $C$  with nonempty values in  $Z$ . The equality*

$$\lim_{y \rightarrow 0, y \in C} \|F(y)\| = 0 \tag{11}$$

holds if and only if there exists a positive constant  $M$  such that

$$\|F(y)\| \leq M\|y\| \quad \text{for } y \in C.$$

In the set of all  $\mathbb{Q}_+$ -homogeneous set-valued functions in  $C$  with nonempty values in  $Z$ , satisfying condition (11) we can introduce the functional

$$\|F\| = \sup_{x \in C, x \neq 0} \frac{\|F(x)\|}{\|x\|}. \tag{12}$$

By Lemma 2.2,  $\|F\|$  is finite. We will call this functional a norm.

**Lemma 2.3** (Theorem 3 in [18], see also Lemma 4 in [16]). *Let  $Y$  be a Banach space,  $Z$  a real, normed linear space and let  $C$  be a convex cone in  $Y$ . Suppose that  $(F_j : j \in J)$  is a family of \*additive, continuous set-valued functions  $F_j: C \rightarrow clb(Z)$ . If  $\text{int}C \neq \emptyset$  and for each  $y \in C$  the set  $\bigcup_{j \in J} F_j(y)$  is bounded in  $Z$ , then there exists a constant  $M \in (0, \infty)$  such that*

$$\sup_{j \in J} \|F_j\| \leq M.$$

We say that a function  $\alpha: [0, \infty) \rightarrow [0, \infty)$  is an  $\alpha$ -function, if  $\alpha(t) > 0$  for  $t \in (0, \infty)$ ,  $\alpha(0) = 0 = \lim_{t \rightarrow 0^+} \alpha(t)$ ,  $\alpha(1) = 1$  and both  $\alpha$  and  $\alpha^*$ , where

$$\alpha^*(t) = \begin{cases} \frac{t}{\alpha(t)} & \text{for } t \in (0, \infty), \\ 0 & \text{for } t = 0, \end{cases}$$

are increasing (cf. [1, p. 182]).

Observe, that the function  $\alpha(t) = t^p$ , where  $p \in (0, 1]$  is an  $\alpha$ -function.

For two  $\alpha$ -functions  $\alpha$  and  $\beta$  we write

$$\alpha \prec \beta \iff \alpha(t) = O(\beta(t)) \quad \text{as } t \rightarrow 0^+.$$

Let  $\alpha$  be an  $\alpha$ -function,  $I = [0, 1]$  and let  $C$  be a convex cone in a real, normed linear space  $Y$ . The set  $H_\alpha(I, C)$  consists, by definition, of all functions  $\varphi: I \rightarrow C$  such that

$$h_\alpha(\varphi) := \sup_{s \in (0, 1]} \frac{\omega(\varphi, s)}{\alpha(s)} < \infty, \quad (13)$$

where

$$\omega(\varphi, s) := \sup\{\|\varphi(x_1) - \varphi(x_2)\| : x_1, x_2 \in I, |x_1 - x_2| \leq s\}$$

(cf. [12]). By  $H_\alpha(I, clb(Z))$  we denote the set of all set-valued functions  $\phi: I \rightarrow clb(Z)$  such that  $h_\alpha(\phi) < \infty$ , where

$$\omega(\phi, s) := \sup\{d(\phi(x_1), \phi(x_2)) : x_1, x_2 \in I, |x_1 - x_2| \leq s\}.$$

Note, that all functions from  $H_\alpha(I, C)$  and from  $H_\alpha(I, clb(Z))$  are continuous. In fact, let us fix  $x_1, x_2 \in I$  and let  $\varphi \in H_\alpha(I, C)$ . We have

$$\|\varphi(x_1) - \varphi(x_2)\| \leq \omega(\varphi, |x_1 - x_2|) \leq h_\alpha(\varphi)\alpha(|x_1 - x_2|). \quad (14)$$

Since  $\alpha$  is continuous at 0, by (13) and (14)  $\varphi$  is uniformly continuous. The same reasoning applies to  $\phi \in H_\alpha(I, clb(Z))$ .

We introduce a metric  $\rho_\alpha$  in the set  $H_\alpha(I, C)$  putting  $\rho_\alpha(\varphi, \bar{\varphi}) = \|\varphi - \bar{\varphi}\|_\alpha$ , where

$$\|\varphi\|_\alpha := \|\varphi(0)\| + h_\alpha(\varphi).$$

In the set  $H_\alpha(I, clb(Z))$  one can define a metric setting

$$d_\alpha(\phi, \bar{\phi}) := d(\phi(0), \bar{\phi}(0)) + \sup_{s \in (0, 1]} \frac{\omega(\phi, \bar{\phi}, s)}{\alpha(s)}, \quad \phi, \bar{\phi} \in H_\alpha(I, clb(Z)),$$

where

$$\omega(\phi, \bar{\phi}, s) := \sup\{d(\phi(x_1) + \bar{\phi}(x_2), \phi(x_2) + \bar{\phi}(x_1)) : x_1, x_2 \in I, |x_1 - x_2| \leq s\}$$

(cf. [17]). It may be checked that  $d_\alpha(\phi, \bar{\phi}) < \infty$  and that  $d_\alpha$  is a metric in  $H_\alpha(I, clb(Z))$  (cf. [6]).

Consider the set

$$\mathcal{L}(C, clb(Z)) := \{A: C \rightarrow clb(Z) : A \text{ is } * \text{ additive and continuous}\}.$$

Since every  $\ast$ additive set-valued function  $A: C \rightarrow clb(Z)$  is  $\mathbb{Q}_+$ -homogeneous, for each  $A \in \mathcal{L}(C, clb(Z))$  we have

$$\|A(y)\| \leq \|A\| \|y\|, \quad y \in C,$$

where  $\|A\|$  is defined by (12). Thus, for  $A, B \in \mathcal{L}(C, clb(Z))$  we have  $d(A(y), B(y)) \leq \|A(y)\| + \|B(y)\| \leq (\|A\| + \|B\|)\|y\|$  and

$$d_{\mathcal{L}}(A, B) := \sup_{y \in C, y \neq 0} \frac{d(A(y), B(y))}{\|y\|}$$

is finite. It is easily seen, that  $d_{\mathcal{L}}$  yields a metric in  $\mathcal{L}(C, clb(Z))$ .

Now let  $\alpha, \beta$  be  $\alpha$ -functions. We will prove (in Theorem 2.8) that a uniformly continuous operator of substitution  $N$  mapping  $H_{\alpha}(I, C)$  into  $H_{\beta}(I, clb(Z))$  has to be generated by a function  $F: I \times C \rightarrow clb(Z)$  of the form

$$F(x, y) = A(x, y) \ast B(x),$$

where  $A(x, \cdot)$  is a  $\ast$ additive continuous set-valued function and  $A(\cdot, y), B$  belong to  $H_{\beta}(I, clb(Z))$ .

**Theorem 2.4.** *Let  $I = [0, 1]$  and  $Y$  be a real normed linear space,  $Z$  a Banach space and let  $C$  be a convex cone in  $Y$ . Assume that  $\gamma: [0, \infty) \rightarrow [0, \infty)$  is continuous at 0,  $\gamma(0) = 0$ , and the superposition operator  $N$  is generated by a set-valued function  $F: I \times C \rightarrow clb(Z)$ .*

(a) *Suppose that  $N$  maps  $H_{\alpha}(I, C)$  into  $H_{\beta}(I, clb(Z))$  and*

$$d_{\beta}(N\varphi, N\bar{\varphi}) \leq \gamma(\|\varphi - \bar{\varphi}\|_{\alpha}), \quad \varphi, \bar{\varphi} \in H_{\alpha}(I, C) \tag{15}$$

*Then there exist functions  $A: I \times C \rightarrow clb(Z)$  and  $B: I \rightarrow clb(Z)$  such that  $A(\cdot, y), B \in H_{\beta}(I, clb(Z))$  for every  $y \in C$ ,  $A(x, \cdot) \in \mathcal{L}(C, clb(Z))$  for every  $x \in I$  and*

$$F(x, y) = A(x, y) \ast B(x), \quad x \in I, y \in C.$$

*Moreover, the inequality*

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(|\bar{x} - x|) \tag{16}$$

*holds for all  $x, \bar{x} \in I$  and  $y_1, y_2 \in C$ .*

(b) *Assume that  $\gamma$  is increasing and the condition  $\frac{1}{\beta} \prec \gamma(\frac{1}{\alpha})$  does not hold. Then the operator  $N$  maps  $H_{\alpha}(I, C)$  into  $H_{\beta}(I, clb(Z))$  and satisfies inequality (15) if and only if the function  $F$  is of the form*

$$F(x, y) = B(x), \quad x \in I, y \in C,$$

*where  $B \in H_{\beta}(I, clb(Z))$ . In this case  $N$  is a constant operator.*

*Proof.* (a) Note, that for a given  $y \in C$  a constant function  $\varphi(t) = y, t \in I$ , belongs to the space  $H_{\alpha}(I, C)$ . Since  $N$  maps  $H_{\alpha}(I, C)$  into  $H_{\beta}(I, clb(Z))$ , we have  $N\varphi = F(\cdot, y) \in H_{\beta}(I, clb(Z))$ . Consequently,  $F(\cdot, y)$  is continuous.

For arbitrarily fixed  $y, \bar{y} \in C$ , take  $\varphi, \bar{\varphi}: I \rightarrow C$  defined by

$$\varphi(t) = y, \quad \bar{\varphi}(t) = \bar{y}, \quad t \in I.$$

Then  $\varphi, \bar{\varphi} \in H_\alpha(I, C)$  and, by the assumption, functions  $N\varphi = F(\cdot, y), N\bar{\varphi} = F(\cdot, \bar{y})$  belong to  $H_\beta(I, clb(Z))$  and

$$\|\varphi - \bar{\varphi}\|_\alpha = \|y - \bar{y}\|.$$

From the definition of the metric  $d_\beta$  there is

$$d(N\varphi(0), N\bar{\varphi}(0)) + \frac{\omega(N\varphi, N\bar{\varphi}, 1)}{\beta(1)} \leq d_\beta(N\varphi, N\bar{\varphi}).$$

Therefore, by (15), for all  $x \in I$ :

$$d(F(0, y), F(0, \bar{y})) + d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)) \leq \gamma(\|y - \bar{y}\|). \tag{17}$$

Since

$$\begin{aligned} d(F(x, y), F(x, \bar{y})) &= d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, \bar{y})) \leq \\ &\leq d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)) + d(F(x, \bar{y}) + F(0, y), F(x, \bar{y}) + F(0, \bar{y})) = \\ &= d(F(0, y), F(0, \bar{y})) + d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)) \end{aligned}$$

(17) shows that

$$d(F(x, y), F(x, \bar{y})) \leq \gamma(\|y - \bar{y}\|) \quad \text{for } x \in I.$$

This inequality, the continuity of  $\gamma$  at 0 and the equality  $\gamma(0) = 0$  imply that  $F$  is continuous with respect to the second variable.

Let us fix  $x, \bar{x} \in I, x < \bar{x}, y_1, y_2, \bar{y}_1, \bar{y}_2 \in C$  and define functions

$$\varphi_i(t) := \begin{cases} y_i & \text{for } 0 \leq t \leq x, \\ \frac{\bar{y}_i - y_i}{\bar{x} - x}(t - x) + y_i & \text{for } x < t < \bar{x} \\ \bar{y}_i & \text{for } \bar{x} \leq t \leq 1 \end{cases}$$

for  $i = 1, 2$ . Obviously,  $\varphi_i(I) \subseteq C$ . We shall prove that  $\varphi_i \in H_\alpha(I, C)$ . It is easily seen that

$$\begin{aligned} \omega(\varphi_i, s) &= \|\bar{y}_i - y_i\| \quad \text{for } \bar{x} - x \leq s \leq 1, \\ \omega(\varphi_i, s) &= \frac{s}{\bar{x} - x} \|\bar{y}_i - y_i\| \quad \text{for } 0 \leq s \leq \bar{x} - x. \end{aligned}$$

Since the function  $t \mapsto \frac{t}{\alpha(t)}$  is increasing

$$\sup_{s \in (0, 1]} \frac{\omega(\varphi_i, s)}{\alpha(s)} = \frac{\|\bar{y}_i - y_i\|}{\alpha(\bar{x} - x)}.$$

Hence,  $\varphi_i \in H_\alpha(I, C)$  and  $\|\varphi_i\|_\alpha = \|y_i\| + \frac{\|\bar{y}_i - y_i\|}{\alpha(\bar{x} - x)}$ . In particular

$$\|\varphi_1 - \varphi_2\|_\alpha = \|y_1 - y_2\| + \frac{\|\bar{y}_1 - \bar{y}_2 - y_1 + y_2\|}{\alpha(\bar{x} - x)}. \tag{18}$$

From (15) and the definition of  $d_\beta$ :

$$\frac{\omega(N\varphi_1, N\varphi_2, \bar{x} - x)}{\beta(\bar{x} - x)} \leq d_\beta(N\varphi_1, N\varphi_2) \leq \gamma(\|\varphi_1 - \varphi_2\|_\alpha)$$

and since  $\varphi_i(x) = y_i$  and  $\varphi_i(\bar{x}) = \bar{y}_i$ ,

$$d(F(x, y_1) + F(\bar{x}, \bar{y}_2), F(\bar{x}, \bar{y}_1) + F(x, y_2)) \leq \gamma(\|\varphi_1 - \varphi_2\|_\alpha)\beta(\bar{x} - x). \tag{19}$$

Taking arbitrary  $u, v \in C$  and putting  $y_1 = \bar{y}_2 = \frac{u+v}{2}$ ,  $\bar{y}_1 = u$ ,  $y_2 = v$  we get

$$\|\varphi_1 - \varphi_2\|_\alpha = \frac{\|u - v\|}{2}$$

and

$$d\left(F\left(x, \frac{u+v}{2}\right) + F\left(\bar{x}, \frac{u+v}{2}\right), F(\bar{x}, u) + F(x, v)\right) \leq \gamma\left(\frac{\|u - v\|}{2}\right)\beta(\bar{x} - x).$$

Letting  $\bar{x}$  tend to  $x$ , since  $\lim_{t \rightarrow 0^+} \beta(t) = 0$ , from the continuity of  $F$  with respect to the first variable we obtain

$$d\left(2F\left(x, \frac{u+v}{2}\right), F(x, u) + F(x, v)\right) = 0,$$

i.e.

$$F\left(x, \frac{u+v}{2}\right) = \frac{1}{2}[F(x, u) \overset{*}{+} F(x, v)]$$

for all  $x \in I$ . This shows that  $F(x, \cdot)$  is  $\overset{*}{J}$ ensen, therefore there exist functions  $A: I \times C \rightarrow clb(Z)$  and  $B: I \rightarrow clb(Z)$  such that  $A(x, \cdot)$  is  $\overset{*}{a}$ dditive for  $x \in I$  and

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C \tag{20}$$

(cf. Lemma 2.1).

To prove that  $A(x, \cdot)$  ( $x \in I$ ) is continuous let us fix  $y, \bar{y} \in C$ . We have

$$d(A(x, y), A(x, \bar{y})) = d(A(x, y) \overset{*}{+} B(x), A(x, \bar{y}) \overset{*}{+} B(x)) = d(F(x, y), F(x, \bar{y})),$$

therefore, the continuity of  $F(x, \cdot)$  implies the continuity of  $A(x, \cdot)$ .

From the  $\overset{*}{a}$ dditivity of  $A(x, \cdot)$  we get  $A(x, 0) = \{0\}$ , whence

$$F(x, 0) = A(x, 0) \overset{*}{+} B(x) = B(x). \tag{21}$$

Since  $F(\cdot, y) \in H_\beta(I, clb(Z))$  for all  $y \in C$ , (21) shows that  $B \in H_\beta(I, clb(Z))$ .

Now we shall prove that  $A(\cdot, y) \in H_\beta(I, clb(Z))$  for every  $y \in C$ . Let us fix  $s \in (0, 1]$ ,  $x, \bar{x} \in I$  such that  $|\bar{x} - x| \leq s$  and  $y \in C$ . Obviously

$$\begin{aligned} d(A(x, y), A(\bar{x}, y)) &= d(A(x, y) + B(x), A(\bar{x}, y) + B(x)) \leq \\ &\leq d(A(x, y) + B(x), A(\bar{x}, y) + B(\bar{x})) + d(A(\bar{x}, y) + B(\bar{x}), A(\bar{x}, y) + B(x)) = \\ &= d(F(x, y), F(\bar{x}, y)) + d(B(x), B(\bar{x})), \end{aligned}$$

whence

$$d(A(x, y), A(\bar{x}, y)) \leq \omega(F(\cdot, y), s) + \omega(B, s)$$

and

$$\frac{\omega(A(\cdot, y), s)}{\beta(s)} \leq h_\beta(F(\cdot, y)) + h_\beta(B).$$

The inequality above shows now, that  $A(\cdot, y) \in H_\beta(I, clb(Z))$  for every  $y \in C$ .

To show (16) take  $x, \bar{x} \in I$  such that  $x \leq \bar{x}$  and  $y_1, y_2 \in C$ . Setting  $\bar{y}_1 = y_1, \bar{y}_2 = y_2$  in (18) and (19) we obtain

$$d(F(x, y_1) + F(\bar{x}, y_2), F(\bar{x}, y_1) + F(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\bar{x} - x).$$

Hence

$$\begin{aligned} d(A(x, y_1) + B(x) + A(\bar{x}, y_2) + B(\bar{x}), A(\bar{x}, y_1) + B(\bar{x}) + A(x, y_2) + B(x)) = \\ = d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\bar{x} - x). \end{aligned}$$

The obtained inequality

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\bar{x} - x)$$

for all  $y_1, y_2 \in C$  and  $x, \bar{x} \in I$ ,  $x < \bar{x}$  is also true in the case when  $x \geq \bar{x}$ , which completes the proof of part (a).

(b) It is sufficient to prove necessity. Setting  $y_1 = y_2$  in (18) and (19) we get

$$d(F(\bar{x}, \bar{y}_1), F(\bar{x}, \bar{y}_2)) \leq \gamma\left(\frac{\|\bar{y}_1 - \bar{y}_2\|}{\alpha(\bar{x} - x)}\right)\beta(\bar{x} - x)$$

for all  $x, \bar{x} \in I$  such that  $x < \bar{x}$  and for all  $\bar{y}_1, \bar{y}_2 \in C$ . In the case  $\|\bar{y}_1 - \bar{y}_2\| \leq 1$  by the monotonicity of  $\gamma$  we have

$$d(F(\bar{x}, \bar{y}_1), F(\bar{x}, \bar{y}_2)) \leq \gamma\left(\frac{1}{\alpha(\bar{x} - x)}\right)\beta(\bar{x} - x). \quad (22)$$

Since the condition  $\frac{1}{\gamma(\frac{1}{\alpha})} < \beta$  does not hold, we can find a sequence  $(t_n), t_n \in (0, 1], t_n \rightarrow 0$ , such that

$$\beta(t_n)\gamma\left(\frac{1}{\alpha(t_n)}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (23)$$

Take  $x \in [0, 1)$  and  $\bar{x}_n := x + t_n$ . Then  $\bar{x}_n \in [0, 1]$  for a large enough  $n$  and  $\bar{x}_n \rightarrow x$ . Since  $F(\cdot, y), y \in C$  is continuous, from (22) and (23) we deduce that  $F(x, \bar{y}_1) = F(x, \bar{y}_2), x \in [0, 1]$  and  $\bar{y}_1, \bar{y}_2 \in C$ .

In the case  $\|\bar{y}_1 - \bar{y}_2\| > 1$ , fix  $n$  large enough to have  $\frac{1}{n}\|\bar{y}_1 - \bar{y}_2\| \leq 1$ . Setting  $y^i = \bar{y}_1 + \frac{i}{n}(\bar{y}_2 - \bar{y}_1), i = 0, 1, \dots, n-1$ , we obtain  $\|y^{i+1} - y^i\| \leq 1$ . By the above, we get

$F(x, y^i) = F(x, y^{i+1})$  for all  $x \in I$  and  $i = 0, 1, \dots, n - 1$ , whence  $F(x, \bar{y}_1) = F(x, \bar{y}_2)$  for all  $x \in I$  and  $\bar{y}_1, \bar{y}_2 \in C$ . In consequence,  $F(x, y) = F(x, 0) =: B(x)$  for  $x \in I$  and  $x \in C$ , which completes the proof.  $\square$

**Remark 2.5.** We denote by  $\mathcal{A}$  the set of all functions  $\varphi \in H_\alpha(I, C)$  of the form

$$\varphi(t) := \begin{cases} y & \text{for } 0 \leq t \leq x, \\ \frac{\bar{y}-y}{\bar{x}-x}(t-x) + y & \text{for } x < t < \bar{x}, \\ \bar{y} & \text{for } \bar{x} \leq t \leq 1 \end{cases}$$

for some  $x, \bar{x} \in I, x < \bar{x}, y, \bar{y} \in C$ . Theorem 2.4 remains true if inequality (15) is assumed only for all  $\varphi, \bar{\varphi} \in \mathcal{A}$ .

**Remark 2.6.** Assuming, that  $\gamma$  in Theorem 2.4 is increasing does not cause any loss of generality.

For a given  $\gamma: [0, \infty) \rightarrow [0, \infty)$  we can take  $\gamma^*: [0, \infty) \rightarrow [0, \infty)$  defined by  $\gamma^*(t) = \sup_{s \in [0, t]} \gamma(s)$ .

**Remark 2.7.** If in Theorem 2.4,  $\gamma(t) = Lt$  (for some constant  $L \geq 0$ ) and the function  $F$  maps  $I \times C$  into the space  $cc(Z)$  of all nonempty, convex and compact subsets of  $Z$ , we can replace  $\overset{*}{+}$  by the usual algebraic sum of two sets and we get the result obtained by J.J. Ludew in [6].

Condition (15) in Theorem 2.4 can be replaced by the uniform continuity of  $N$ .

**Theorem 2.8.** *Let  $Y$  be a real normed linear space,  $Z$  a Banach space and  $C$  a convex cone in  $Y$ . Suppose that the superposition operator  $N$  of the generator  $F: I \times C \rightarrow clb(Z)$  maps  $H_\alpha(I, C)$  into  $H_\beta(I, clb(Z))$  and that  $N$  is uniformly continuous. Then there exist functions  $A: I \times C \rightarrow clb(Z)$  and  $B: I \rightarrow clb(Z)$  such that  $A(\cdot, y), B \in H_\beta(I, clb(Z))$  for every  $y \in C, A(x, \cdot) \in \mathcal{L}(C, clb(Z))$  for every  $x \in I$  and*

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C.$$

*Proof.* Suppose that  $N$  is uniformly continuous. Then for every  $\varepsilon > 0$  there is  $\delta > 0$  such that for all  $\varphi, \bar{\varphi} \in H_\alpha(I, C)$

$$\|\varphi - \bar{\varphi}\|_\alpha \leq \delta \implies d_\beta(N\varphi, N\bar{\varphi}) \leq \varepsilon.$$

Let  $\gamma: [0, \infty) \rightarrow [0, \infty)$  be defined by

$$\gamma(t) := \sup\{d_\beta(N\varphi, N\bar{\varphi}) : \|\varphi - \bar{\varphi}\|_\alpha \leq t\}, \quad t \geq 0.$$

The function  $\gamma$  is well defined. Indeed, fix  $\delta > 0$  such that for all  $\varphi, \bar{\varphi} \in H_\alpha(I, C)$

$$\|\varphi - \bar{\varphi}\|_\alpha \leq \delta \implies d_\beta(N\varphi, N\bar{\varphi}) \leq 1. \tag{24}$$

Therefore, we have  $\gamma(t) \leq 1$  for all  $t \in [0, \delta]$ . Take  $t \geq 0, s \geq 0, t + s > 0$  and  $\varphi, \bar{\varphi} \in H_\alpha(I, C)$  such that  $\|\varphi - \bar{\varphi}\|_\alpha \leq t + s$ . The function  $\psi = \frac{t}{t+s}\varphi + \frac{s}{t+s}\bar{\varphi}$  also belongs to  $H_\alpha(I, C)$  and

$$\|\varphi - \psi\|_\alpha = \frac{s}{t+s} \|\varphi - \bar{\varphi}\|_\alpha \leq s, \quad \|\psi - \bar{\varphi}\|_\alpha = \frac{t}{t+s} \|\varphi - \bar{\varphi}\|_\alpha \leq t.$$

Thus, by the definition of  $\gamma$

$$d_\beta(N\varphi, N\bar{\varphi}) \leq d_\beta(N\varphi, N\psi) + d_\beta(N\psi, N\bar{\varphi}) \leq \gamma(s) + \gamma(t)$$

and consequently

$$\gamma(s+t) \leq \gamma(s) + \gamma(t).$$

In particular,  $\gamma(2t) \leq 2\gamma(t)$ , whence by induction we obtain

$$\gamma(nt) \leq n\gamma(t) \tag{25}$$

for all  $n \in \mathbb{N}$  and  $t \geq 0$ . For a given  $t \geq 0$  there exists a positive integer  $n$  such that  $\frac{t}{n} < \delta$ . From (24) and (25) it follows that

$$\gamma(t) = \gamma\left(n \frac{t}{n}\right) \leq n\gamma\left(\frac{t}{n}\right) \leq n < \infty.$$

Since  $N$  is uniformly continuous,  $\gamma$  is continuous at 0,  $\gamma(0) = 0$  and obviously

$$d_\beta(N\varphi, N\bar{\varphi}) \leq \gamma(\|\varphi - \bar{\varphi}\|_\alpha), \quad \varphi, \bar{\varphi} \in H_\alpha(I, C),$$

the result is a consequence of Theorem 2.4. □

The following result may be proved in the same way as Lemma 5 in [16].

**Lemma 2.9.** *Let  $Y$  and  $Z$  be two real, normed linear spaces and  $C$  a convex cone in  $Y$  with nonempty interior. Then there exists a positive constant  $M_0$  such that for every continuous, \*additive, set-valued function  $F: C \rightarrow clb(Z)$  the inequality*

$$d(F(x), F(y)) \leq M_0 \|F\| \|x - y\|, \quad x, y \in C$$

holds.

The following theorem is a converse of part (a) of Theorem 2.4.

**Theorem 2.10.** *Let  $Y$  be a Banach space,  $Z$  a real normed linear space,  $C$  a convex cone in  $Y$  with nonempty interior and let  $\alpha, \beta$  be two  $\alpha$ -functions such that  $\alpha \prec \beta$ . Assume that  $A(\cdot, y), B \in H_\beta(I, clb(Z))$  for  $y \in C$  and  $A(x, \cdot) \in \mathcal{L}(C, clb(Z))$  for  $x \in I$ . Moreover, assume that for some increasing, continuous at 0 function  $\gamma: [0, \infty) \rightarrow [0, \infty)$ , such that  $\gamma(0) = 0$ , the inequality*

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(|\bar{x} - x|) \tag{26}$$

holds for all  $x, \bar{x} \in I$  and  $y_1, y_2 \in C$ . If a set-valued function  $F: I \times C \rightarrow clb(Z)$  is of the form

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C,$$

then the operator of substitution  $N$  generated by  $F$  maps the set  $H_\alpha(I, C)$  into the set  $H_\beta(I, clb(Z))$  and satisfies inequality (15) with a function  $\gamma_1$ , where  $\gamma_1(t) = c(t + \gamma(t))$ ,  $t \geq 0$  and  $c$  is a constant.



*Proof.* First, we will prove that the set  $\bigcup_{x \in I} A(x, y)$  is bounded for an arbitrary  $y \in C$ . Let  $x \in I, y \in C$ . We have

$$\begin{aligned} \|A(x, y)\| &= d(A(x, y), \{0\}) \leq d(A(x, y), A(0, y)) + d(A(0, y), \{0\}) = \\ &= d(A(x, y), A(0, y)) + \|A(0, y)\|. \end{aligned}$$

Moreover, since  $A(\cdot, y) \in H_\beta(I, clb(Z))$ ,

$$d(A(x, y), A(0, y)) \leq \omega(A(\cdot, y), 1) = \frac{\omega(A(\cdot, y), 1)}{\beta(1)} \leq h_\beta(A(\cdot, y)) < \infty.$$

Hence

$$\|A(x, y)\| \leq h_\beta(A(\cdot, y)) + \|A(0, y)\|, \quad x \in I.$$

Since  $\{A(x, \cdot)\}_{x \in I}$  is a family of \*additive and continuous functions, by Lemma 2.3 there exists a constant  $M > 0$  such that

$$\sup_{x \in I} \|A(x, y)\| \leq M\|y\|, \quad y \in C.$$

Hence, and by Lemma 2.9, we deduce that

$$d(A(x, y), A(x, \bar{y})) \leq M_0 M \|y - \bar{y}\| \tag{27}$$

for all  $x \in I$  and  $y, \bar{y} \in C$ .

We shall prove now that  $N$  maps  $H_\alpha(I, C)$  into  $H_\beta(I, clb(Z))$ . Let  $\varphi \in H_\alpha(I, C)$  and  $x, \bar{x} \in I$ . The inequality

$$d(A(x, y), A(\bar{x}, y)) \leq \gamma(\|y\|)\beta(|\bar{x} - x|) \tag{28}$$

is a consequence of (26). From (27) and (28) we obtain

$$\begin{aligned} d(N\varphi(x), N\varphi(\bar{x})) &= d(A(x, \varphi(x)) + B(x), A(\bar{x}, \varphi(\bar{x})) + B(\bar{x})) \leq \\ &\leq d(A(x, \varphi(x)), A(\bar{x}, \varphi(\bar{x}))) + d(B(x), B(\bar{x})) \leq \\ &\leq d(A(x, \varphi(x)), A(\bar{x}, \varphi(x))) + d(A(\bar{x}, \varphi(x)), A(\bar{x}, \varphi(\bar{x}))) + d(B(x), B(\bar{x})) \leq \\ &\leq \gamma(\|\varphi(x)\|)\beta(|\bar{x} - x|) + M_0 M \|\varphi(x) - \varphi(\bar{x})\| + d(B(x), B(\bar{x})) \end{aligned}$$

for all  $x, \bar{x} \in I$ . Since

$$\|\varphi(x)\| \leq \|\varphi(0)\| + \frac{\|\varphi(x) - \varphi(0)\|}{\alpha(x - 0)} \alpha(x - 0), \quad x \in (0, 1]$$

we have  $\|\varphi(x)\| \leq \|\varphi\|_\alpha$  for every  $x \in I$ . Now take  $s \in (0, 1]$  and  $x, \bar{x} \in I$  such that  $|x - \bar{x}| \leq s$ . The monotonicity of  $\gamma$  and  $\beta$  implies, that

$$d(N\varphi(x), N\varphi(\bar{x})) \leq \gamma(\|\varphi\|_\alpha)\beta(s) + M_0 M \omega(\varphi, s) + \omega(B, s).$$

Therefore, for every  $s \in (0, 1]$  we obtain

$$\begin{aligned} \frac{\omega(N\varphi, s)}{\beta(s)} &\leq \gamma(\|\varphi\|_\alpha) + M_0M \frac{\omega(\varphi, s)}{\alpha(s)} \frac{\alpha(s)}{\beta(s)} + \frac{\omega(B, s)}{\beta(s)} \leq \\ &\leq \gamma(\|\varphi\|_\alpha) + LM_0Mh_\alpha(\varphi) + h_\beta(B), \end{aligned}$$

where  $L > 1$  is a constant such that  $\frac{\alpha(s)}{\beta(s)} \leq L$ ,  $s \in (0, 1]$  (by the assumption  $\alpha \prec \beta$ ). Thus,  $h_\beta(N\varphi) < \infty$  and  $N\varphi \in H_\beta(I, clb(Z))$ .

What remains to show is the fact, that  $N$  satisfies (15). Let  $\varphi, \bar{\varphi} \in H_\alpha(I, C)$ ,  $s \in (0, 1]$  and take  $x, \bar{x} \in I$  such that  $|\bar{x} - x| \leq s$ . Inequalities (27) and (26) imply that

$$\begin{aligned} d(N\varphi(x) + N\bar{\varphi}(\bar{x}), N\varphi(\bar{x}) + N\bar{\varphi}(x)) &= \\ &= d(A(x, \varphi(x)) + A(\bar{x}, \bar{\varphi}(\bar{x})), A(\bar{x}, \varphi(\bar{x})) + A(x, \bar{\varphi}(x))) = \\ &= d\left(A(x, \varphi(x) + \bar{\varphi}(\bar{x})) + A(\bar{x}, \bar{\varphi}(\bar{x})) + A(x, \varphi(\bar{x})), \right. \\ &\quad \left. A(x, \varphi(\bar{x}) + \bar{\varphi}(x)) + A(x, \bar{\varphi}(\bar{x})) + A(\bar{x}, \varphi(\bar{x}))\right) \leq \\ &\leq d(A(x, \varphi(x) + \bar{\varphi}(\bar{x})), A(x, \varphi(\bar{x}) + \bar{\varphi}(x))) + \\ &\quad + d(A(\bar{x}, \bar{\varphi}(\bar{x})) + A(x, \varphi(\bar{x})), A(x, \bar{\varphi}(\bar{x})) + A(\bar{x}, \varphi(\bar{x}))) \leq \\ &\leq M_0M\|(\varphi - \bar{\varphi})(x) - (\varphi - \bar{\varphi})(\bar{x})\| + \gamma(\|(\varphi - \bar{\varphi})(\bar{x})\|)\beta(|\bar{x} - x|) \leq \\ &\leq M_0M\omega(\varphi - \bar{\varphi}, s) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha)\beta(s). \end{aligned}$$

Hence

$$\frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \leq M_0M \frac{\omega(\varphi - \bar{\varphi}, s)}{\alpha(s)} \frac{\alpha(s)}{\beta(s)} + \gamma(\|\varphi - \bar{\varphi}\|_\alpha)$$

and therefore

$$\sup_{s \in (0, 1]} \frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \leq M_0MLh_\alpha(\varphi - \bar{\varphi}) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha).$$

In consequence

$$\begin{aligned} d_\beta(N\varphi, N\bar{\varphi}) &= d(N\varphi(0), N\bar{\varphi}(0)) + \sup_{s \in (0, 1]} \frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \leq \\ &\leq M_0M\|(\varphi - \bar{\varphi})(0)\| + M_0MLh_\alpha(\varphi - \bar{\varphi}) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha) \leq \\ &\leq M_0ML\|\varphi - \bar{\varphi}\|_\alpha + \gamma(\|\varphi - \bar{\varphi}\|_\alpha). \end{aligned}$$

Setting  $c = \max\{1, M_0ML\}$  and  $\gamma_1(t) = c(t + \gamma(t))$  we get

$$d_\beta(N\varphi, N\bar{\varphi}) \leq \gamma_1(\|\varphi - \bar{\varphi}\|_\alpha).$$

This finishes the proof. □

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